A note on the monotonicity and superadditivity of TU cooperative games

Jean-François Caulier

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Jean-François Caulier
Facultés Universitaires Saint-Louis
43 Bd du jardin Botanique,1000 Bruxelles, Belgium
tel. : +32-2-211-79-44.

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Abstract

In this note we make a comparison between the class of monotonic TU cooperative games and the class of superadditive TU cooperative games. We first provide the equivalence between a weakening of the class of superadditive TU games and zero-monotonic TU games. Then, we show that zero-monotonic TU games and monotonic TU games are different classes. Finally, we show under which restrictions the classes of superadditive and monotonic TU games can be related.

Keywords : TU cooperative games; superadditivity; monotonicity

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1 Introduction

To the best of our knowledge, the general relationship between the class of *monotonic* TU cooperative games (*TU games* for short) and the class of *superadditive* TU games appears to be missing in the literature. Usually, cooperative games textbooks provide the definitions of both classes in sequence without any mention to the conditions under which they can be related.[1] Ignoring the (simple) conditions under which one class is included in the other class mistakenly led some authors to restrict redundantly the games they want to study to the class of monotonic **and** superadditive TU games (e.g. see Laruelle and Valenciano [2] p.45). However, Maschler, Peleg and Shapley [3] (p.309) note that every superadditive TU game is *zero-monotonic*. A TU game is zero-monotonic when the (unique) corresponding zero-normalized TU game (where the worth of any singleton is zero) is monotonic. Other noticeable exceptions appear in Weber [7] who states that no class contains the other one, and in Jaffray and Mongin [1] where they state that if the TU game is everywhere non-negative, then the class of superadditive TU games is contained in the class of monotonic TU games. But as far as we know, no formal proof of these statements has been provided. In order to prove these statements, we first define formally the classes under study and relate them to derived concepts (*weakly-superadditive* and *zero-monotonic* TU games).

2 Notation and definitions

A **cooperative game with transferable utility** (TU game) is a pair (N,v) where $N=\{1,\dots,n\}$ is a finite set of players and v is a function associating a real value $v(S)$ to each subset S of N such that $v(\emptyset)=0$.

Each subset S of N is called a *coalition* and the set of possible coalitions on N is denoted by 2^N. The function $v:2^N\rightarrow\mathbb{R}$ is called the *characteristic function* of the game (N,v) and $v(S)$ is the worth achieved by the members of coalition S in the game (N,v). In TU games, the worth of a coalition can be redistributed among its members in any possible way.
We denote by $\mathcal{G}^N$ the set of possible TU games on N. A particular class of TU games among $\mathcal{G}^N$ is the class of *superadditive* TU games :

Definition 1. *A TU game (N,v) is superadditive if and only if*

$$v(S)+v(T)\leq v(S\cup T)$$

for any $S\subset N$, $T\subset N$ and $S\cap T=\emptyset$.

[1] See Peleg and Sudhölter [6] and references therein.

Alternatively (see Moulin [4]), take any collection $S_1, \ldots, S_K$ of pairwise disjoint subsets of N whose union is N, then a game v is superadditive (or *cohesive* in Osborne and Rubinstein [5]) if

$$\sum_{k=1}^{K} v(S_k) \leq v(N).$$

Intuitively, the condition of superadditivity states that players can collectively achieve a higher value than in separated coalitions. This condition is reminiscent to increasing return of scales : two coalitions can jointly do at least as better as they do separately.

We denote the set of superadditive TU games on N by $\mathscr{G}_s^N$.

Definition 2. *A TU game (N,v) is monotonic if and only if $S \subset T \Rightarrow v(S) \leq v(T)$, for any $S,T \subset N$.*

This condition simply states that the bigger the coalition, the higher its value.

We denote the set of monotonic TU games on N by $\mathscr{G}_m^N$.

It is quite easy to design an example of monotonic TU game which violates superadditivity : the worth of any two distinct coalitions put together might be greater than the worth of each coalition taken individually, but might be smaller than the sum of their worths. Besides, the following example presents a superadditive TU game violating monotonicity.

Example 1. *Suppose that three individuals living in the same neighborhood want to link up to the local sewer system. The linking cost for the citizen A alone is 1 whereas it is 5 for citizen B and 5 for citizen C. The cost of building and linking up two neighbors is 6 and if the three neighbors are served collectively the cost is 7. Each citizen receives a state subsidy of 4 to link up to the sewer system. This situation can be described by the following characteristic function :*

$$v(\{A\}) = 3, v(\{B\}) = v(\{C\}) = -1,$$

$$v(\{A,B\}) = v(\{A,C\}) = v(\{B,C\}) = 2,$$

$$v(\{A,B,C\}) = 5$$

Since $v(\{A\}) > v(\{A,B\})$, the game is not monotonic. Now, $v(\{i\}) + v(\{j\}) \leq v(\{i,j\})$ for any $i \neq j$ and $v(\{i,j\}) + v(\{k\}) \leq v(\{A,B,C\})$ for any $i \neq j \neq k$, then the game is superadditive.

3 Classes of TU games related to superadditive TU games

Definition 3. *Let (N,v) be a TU game. We say that v is* **essential superadditive** *if*

$$v(S) \geq \sum_{k} v(S^k)$$

for any $S \subset N$ and $(S^k)_k$ any partition of S.

The idea of essential superadditivity (see e.g. Wooders [8]) is that "*an option open to a group of players is to cooperate only within elements of a partition of the group.*"

Definition 4. *A TU game (N,v) is* **essential** *if*

$$v(N) \geq \sum_{i \in N} v(\{i\}).$$

Note that the definition of essential TU games is built into the definition of essential superadditivity by taking $S = N$ and $(S^k)_k$ the singleton partition.

We denote the class of essential superadditive games on N by $\mathscr{G}_{es}^N$.

In the next proposition we show the equivalence between the concepts of essential superadditivity and superadditivity.

Proposition 1. *Let (N,v) be a TU game. Then (N,v) is superadditive if and only if (N,v) is essential superadditive :*

$$(N,v) \in \mathscr{G}_{es}^N \Leftrightarrow (N,v) \in \mathscr{G}_{s}^N.$$

Proof.

1. We first prove that $\mathscr{G}_{es}^N \subset \mathscr{G}_{s}^N$.

   Suppose that (N,v) is essential superadditive. Define $S = A \cup B$ such that $A \cap B = \emptyset$. Since $\{A,B\}$ is a partition of S, by essential superadditivity we have $v(A) + v(B) \leq v(S)$.

2. To prove that $\mathscr{G}_{s}^N \subset \mathscr{G}_{es}^N$, we show that if $(N,v) \in \mathscr{G}_{s}^N$, then (N,v) must be in $\mathscr{G}_{es}^N$.

   Suppose that $(N,v) \in \mathscr{G}_{s}^N$. Then for any $S \subset N$, $v(A) + v(B) \leq v(S)$ for any $A, B \subset S$, $A \cap B = \emptyset$ and $A \cup B = S$. In the following we iterate the following steps:

Step 1: If $(N,v) \notin \mathscr{G}^N_{es}$ then there exists a partition $\{S_k\}_{k\in K}$ of S such that $v(S) < \sum_{k\in K} v(S_k)$.

Step 2: Choose $A, B \subset S$ compatible with the partition, i.e. $A \cap B = \emptyset, A \cup B = S$ and

$$\bigcup_{k\in K'} S_k = A$$

$$\bigcup_{k\in K''} S_k = B$$

with $K' \cap K'' = \emptyset$, $K' \cup K'' = K$. Then either

$$\sum_{k\in K'} v(S_k) > v(A) \tag{1}$$

or

$$\sum_{k\in K''} v(S_k) > v(B) \tag{2}$$

Note that both (1) and (2) cannot be violated at the same time since otherwise it would violate superadditivity.

Without loss of generality, suppose that equation (1) holds. We now iterate the procedure and apply the step 1 to the set A.

Step 1: There exists a partition of A such that $v(A) < \sum_{k\in K'''} v(A_k)$. Such a partition must exist by equation (1) (indeed take $K''' = K'$).

Step 2: Choose $C, D \subset A$ compatible with the partition $\{A_k\}_{k\in K'''}$.

Continue the procedure until you find a set T such that in step 1 we have $v(T) < v(T_1) + v(T_2)$ with $T_1, T_2 \subset T$, $T_1 \cap T_2 = \emptyset$ and $T_1 \cup T_2 = T$, contradicting the superadditivity of (N,v). By the finiteness property of the player set N, the procedure eventually stops after a finite number of iterations.

□

In some cases, we are interested in extending the class of superadditive TU games by weakening the definition of superadditivity in the following way :

Definition 5. *Let (N,v) be a TU game. Then (N,v) is* **weakly superadditive** *if*

$$v(S) + v(\{i\}) \leq v(S \cup \{i\})$$

for all $S \subset N \setminus \{i\}$.

We denote the class of weakly superadditive TU games on N by $\mathscr{G}^N_{ws}$.

Proposition 2. *Let (N, v) be a TU game. Then*

$$(N, v) \in \mathscr{G}_s^N \Rightarrow (N, v) \in \mathscr{G}_{ws}^N.$$

Proof.

1. We can show directly that any superadditive TU game is also weakly superadditive. Consider any $(N, v) \in \mathscr{G}_s^N$. Take $S \subset N \setminus \{i\}$, then

   $$v(S) + v(\{i\}) \leq v(S \cup \{i\})$$

   which is the desired conclusion.

2. We now show that a TU game can be weakly superadditive while not being superadditive. Consider $N = \{1, 2, 3, 4\}$ and v such that $v(\{i\}) = 0.1$ for any $i \in N$, $v(S) = 1$ for $|S| = 2$, $v(S) = 1.2$ for $|S| = 3$ and $v(N) = 1.5$.

   Then for any $S \subset N$,

   $$v(S) + v(\{i\}) \leq v(S \cup \{i\})$$

   but $v(S) + v(T) > v(S \cup T)$ for any $S, T \subset N$, $S \cap T = \emptyset$ such that $|S| = |T| = 2$.

□

4 Classes of TU games related to monotonic TU games

In many applications of TU cooperative games, it is assumed that the worth achieved by an isolated player is zero. By the following definition, we show that any TU game (N, v) can be normalized such that the worth of any singleton is zero.

Definition 6. *A TU game (N, v) is* **zero normalized** *if and only if*

$$v(S) = v(S) - \sum_{i \in S} v(\{i\})$$

for all $S \subset N$.

That is if $v(\{i\}) = 0$ for all $i \in N$.

Definition 7. *A TU game (N, v) is* **zero-monotonic** *if and only if*

$$v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$$

for all $S \subset T \subset N$.

That is (N,v) is zero-monotonic if its (unique) zero normalization is monotonic : for all $S \subset T \subset N$, $v(S) - \sum_{i \in S} v(\{i\}) \leq v(T) - \sum_{i \in T} v(\{i\})$

$$\Leftrightarrow v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T).$$

We denote the class of zero-monotonic TU games on N by $\mathscr{G}^N_{0m}$. The next proposition that shows that the classes of zero-monotonic and monotonic TU games are not equivalent will be useful later in the study of the relationship between monotonic and superadditive TU games.

Proposition 3. *The class of monotonic TU games $(N,v) \in \mathscr{G}^N \setminus \mathscr{G}^N_{0m}$ and the class of zero-monotonic TU games $(N,v) \in \mathscr{G}^N \setminus \mathscr{G}^N_m$ are non-empty.*

Proof.

1. We first show the existence of zero-monotonic TU games that are not monotonic. If (N,v) is not monotonic, then there exists $S \subset N$ such that $v(S) > v(T)$ for $S \subset T \subset N$. Suppose now that $v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$, then v is zero-monotonic.

   Example 2. *Let $N = \{1,2\}$, and v be such that $v(\{1\}) = 2$, $v(\{2\}) = -1$ and $v(\{1,2\}) = 1.5$.*

2. We now provide an example of monotonic TU game which is not zero-monotonic.

   Example 3. *Let $N = \{1,2,3\}$, and v be such that $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,2\}) = 1$, $v(\{1,3\}) = v(\{2,3\}) = 2$ and $v(\{1,2,3\}) = 2.5$. We can see that v is monotonic. We now show that its zero-normalization is not monotonic. Let $w(S) = v(S) - \sum_{i \in S} v(\{i\})$. Then $w(\{1\}) = w(\{2\}) = w(\{3\}) = 0$, $w(\{1,2\}) = -1, v(\{1,3\}) = v(\{2,3\}) = 0$ and $v(\{1,2,3\}) = -0.5$, so that w is not monotonic.*

$\square$

We now state the perfect coincidence between the class of weakly superadditive TU games and zero-monotonic TU games.

Proposition 4. *Let* (N, v) *be a TU game. Then,* (N, v) *is weakly superadditive if and only if* (N, v) *is zero-monotonic.*

Proof.

1. Consider (N, v) a zero-monotonic TU game. Take $T = S \cup \{i\}$ with $S \in N \setminus \{i\}$. Then $v(S) + v(\{i\}) \leq v(S \cup \{i\})$, establishing the weak-superadditivity of v.

2. Now, let (N, v) be a weakly superadditive TU game. We want to prove that for any $S \subset N : v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$ whenever $S \subset T$.

   Suppose that $S \subset T$ and let $\{i_1, \ldots, i_K\}$ be a sequence of players covering $T \setminus S$. By weak-superadditivity :

   $$v(S) + v(\{i_1\}) \leq v(S \cup \{i_1\}).$$

   By weak-superadditivity again,

   $$v(S \cup \{i_1\}) + v(\{i_2\}) \leq v(S \cup \{i_1\} \cup \{i_2\}).$$

   Applying weak-superadditivity $K - 2$ times :

   $$v(S \cup \{i_1, \ldots, i_{K-1}\}) + v(\{i_K\}) \leq v(T).$$

   Adding all these inequalities, we get

   $$v(S) + \sum_{k=1}^{K} v(\{i_k\}) \leq v(T).$$

   Because the sequence $\{i_k\}_{k=1}^K$ covering $T \setminus S$ we have chosen is arbitrary, we get the desired conclusion.

□

5 Relations between the classes of monotonic and superadditive TU games

To show the general relationship between the class of monotonic TU games and the class of superadditive TU games, the following proposition is useful.

Proposition 5. *If (N,v) is a superadditive TU game, then (N,v) is a zero-monotonic TU game.*

Proof. Combine Propositions 2 and 4. □

Theorem 1. *The classes of superadditive TU games $(N,v) \in \mathscr{G}^N \setminus \mathscr{G}_m^N$ and of monotonic TU games $(N,v) \in \mathscr{G}^N \setminus \mathscr{G}_s^N$ are non-empty.*

Proof. Combine Propositions 3 and 5. □

Despite the absence of general relationship between superadditive and monotonic games, there exists a nice relationship between both classes if we restrain the class of superadditive TU games to the class of **nonnegative** superadditive TU games :

Proposition 6. *If (N,v) is any nonnegative superadditive TU game then (N,v) is a monotonic TU game.*

Proof. Choose any TU game $(N,v) \in \mathscr{G}_s^N$ such that $v(S) \geq 0$ for any $S \subset N$.

By superadditivity, $v(S) \leq v(T \cup S)$ for any T,S such that $S \cap T = \emptyset$. Since $S \subset T \cup S$, it implies monotonicity. Hence $\mathscr{G}_s^N \subset \mathscr{G}_m^N$.

To prove that the converse does not hold, it is sufficient to find an example of monotonic TU game violating superadditivity : Choose $N = \{1,2,3\}$ and v such that $v(S) = 1$ for any $S \subset N$ such that $|S| = 1$, $v(S) = 1.5$ for any $S \subset N$ such that $|S| = 2$ and $v(N) = 2.4$. Then (N,v) is monotonic but not superadditive since $v(\{1\}) + v(\{2,3\}) = 2.5 \geq 2.4$.

□

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